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## A Note on Products of Infinite Cyclic Groups.

B. GOLDSMITH (\*)

### Introduction.

In his book [2], Fuchs introduces the notion of a subgroup  $X$  of a Specker group  $P$  being a *product* and goes on to establish a Lemma [2, Lemma 95.1] which yields a useful characterization of the quotient  $P/X$  and enables an easy derivation of Nunke's characterization of epimorphic images of the Specker group [4]. Unfortunately this Lemma is incorrect as we show in section 1. In section 2 by suitably strengthening the hypothesis we regain a characterization of the quotient. Throughout, all groups are additively written Abelian groups and our notation follows the standard works of Fuchs [1], [2].

§ 1. Suppose  $P = \prod_{n=1}^{\infty} \langle e_n \rangle$  is a Specker group, then Fuchs defines a subgroup  $X$  of  $P$  to be a *product*  $\prod_{n=1}^{\infty} \langle x_n \rangle$  if for every  $m$ , the  $m$ -th coordinates of almost all  $x_n$  are 0 and  $X$  consists of all the formal sums  $\sum s_n x_n$ . To avoid confusion with the usual meaning of product (i.e.  $X$  is a product if it is isomorphic to a cartesian product of infinite cyclic groups) we denote a *product* (in the sense of Fuchs) by  $\sum^*$  and reserve the symbol  $\prod$  for the more usual meaning.

LEMMA 1. If  $Y$  is an endomorphic image of  $P$  then  $Y$  is a *product* (in the sense of Fuchs).

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PROOF. Let  $\alpha: P \rightarrow P$  be an endomorphism with  $\text{Im } \alpha = Y$ . Let  $\pi_n$  denote the projection of  $P$  onto  $\langle e_n \rangle$ , and set, for each  $n$ ,  $y_n = e_n \alpha$ . Since  $\langle e_n \rangle$  is slender, the map  $\alpha \pi_n: P \rightarrow \langle e_n \rangle$  maps almost all  $e_i$  to zero i.e.  $y_i \pi_n = e_i \alpha \pi_n = 0$  for all but a finite number of indices  $i$ . Thus for every  $n$ , the  $n$ -th co-ordinates of almost all  $y_i$  are zero and so the set of sums  $\{\sum s_i y_i\}$  is a *product*  $\sum_{i=1}^{\infty} * \langle y_i \rangle$  in  $P$ .

Now define  $\beta: P \rightarrow P$  by  $(\dots, n_i e_i, \dots) \beta = \sum n_i y_i$ . For each  $n = 1, 2, \dots$ ,  $\alpha \pi_n$  and  $\beta \pi_n$  map  $P$  into a slender group and agree on  $S = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$ . Hence  $\alpha \pi_n = \beta \pi_n$  for all  $n$ . So  $\alpha = \beta$  and  $Y = \text{Im } \alpha = \text{Im } \beta = \sum_{i=1}^{\infty} * \langle y_i \rangle$ , and thus  $Y$  is a *product* (in the sense of Fuchs).

COUNTER-EXAMPLE. With  $P = \prod_{i=1}^{\infty} \langle e_i \rangle$ , set  $Y = \prod_{i=1}^{\infty} \langle 2^i e_i \rangle$ . Then  $P/Y \cong \prod_{i=1}^{\infty} \mathbb{Z}(2^i)$  and this is a complete module over the ring  $J_2$  of 2-adic integers. Moreover the torsion submodule of this quotient is not dense in the 2-adic topology. Hence it has a direct summand  $H \cong J_2$  and if  $\langle x \rangle$  is dense in  $H$  then  $H/\langle x \rangle$  is divisible. Choose  $y \in P$  such that  $y$  maps onto  $x$  modulo  $Y$  and let  $X = \langle y, Y \rangle$ . Then certainly  $X$  is isomorphic to  $P$  and hence is an endomorphic image of  $P$ . By Lemma 1  $X$  is a *product* (in the sense of Fuchs). However  $P/X \cong \prod_{i=1}^{\infty} \mathbb{Z}(2^i)/\langle x \rangle$  which contains the divisible subgroup  $H/\langle x \rangle$ . However if the conclusion of Lemma 95.1 in [2] were correct then  $P/X$  would be reduced. So  $X$  is clearly a counter-example to the quoted Lemma.

*Acknowledgement.* The above arguments arose from interesting discussions with Peter Neumann and Adolf Mader. The main idea in the counter-example is essentially due to the former.

§ 2. In this section by introducing an appropriate topological concept we can regain some information about quotients. Let  $P = \prod_{i=1}^{\infty} \langle e_i \rangle$  and topologize  $P$  with the product topology of the discrete topology on each component. We refer to this topology simply as the product topology on  $P$ . The subgroups  $P_n = \prod_{i=n}^{\infty} \langle e_i \rangle$  are a basis of neighbourhoods of zero.

PROPOSITION 2. If  $X$  is a subgroup of  $P$  which is closed in the product topology then

(i)  $X$  is a product  $\sum_{i=1}^{\infty} * \langle x^i \rangle$ ;

(ii)  $P/X$  is isomorphic to a cartesian product of cyclic groups.

PROOF. Part (i) is a well-known result due to Nunke [3]. He shows that there are elements  $x^n$  in  $X$  with (a)  $x_i^n = 0$  for  $i < n$ ; (b)  $x_n^n = 0$  if and only if  $x^n = 0$ ; (c)  $x_n^n$  divides  $u_n$  for all  $u$  in  $X \cap P_n$ . (Subscripts denote components in the product  $P$ .) Moreover if  $X$  is closed,  $X = \sum * \langle x^n \rangle$ .

In establishing (ii) we let  $d_n = x_n^n$  in order to simplify notation. Notice that it follows easily from the properties (a), (b), (c) that if  $X = \sum * \langle y^n \rangle$  also, then  $y_n^n = d_n$  and  $x_{n+1}^n - y_{n+1}^n$  is a multiple of  $d_{n+1}$ .

Suppose  $a \in P$  is given by  $a = (a_1, a_2, \dots)$  then we may write

$$a_1 = r_1 d_1 + s_1 \quad \text{where} \quad 0 \leq s_1 < d_1.$$

Also

$$a_2 - r_1 x_2^1 = r_2 d_2 + s_2 \quad \text{where} \quad 0 \leq s_2 < d_2,$$

$$a_3 - r_1 x_3^1 - r_2 x_3^2 = r_3 d_3 + s_3 \quad \text{where} \quad 0 \leq s_3 < d_3, \quad \text{etc.}$$

Define a map  $\varphi$  from  $P$  onto the cartesian product of the cyclic groups of order  $d_i$  by  $\varphi(a) = (s_1, s_2, \dots, s_n, \dots)$ . We must verify that  $\varphi$  is a well-defined homomorphism. Suppose  $X = \sum * \langle y^n \rangle$  then since  $y_1^1 = x_1^1 = d_1$  we get that  $r_1$  and  $s_1$  are uniquely defined.

Now  $a_2 - r_1 y_2^1 = a_2 - r_1(x_2^1 + k d_2) = (r_2 - r_1 k) d_2 + s_2$  (some  $k \in \mathbb{Z}$ ) and so  $s_2$  is defined as before. Note that  $x^1 - y^1 - k x^2 \in X$  and so by property (c)  $x_3^1 - y_3^1 - k x_3^2$  is a multiple of  $d_3$ . Making this substitution one easily obtains that  $a_3 - r_1 y_3^1 - (r_2 - r_1 k) y_3^2 \equiv s_3 \pmod{d_3}$  and so  $s_3$  is defined as before. Repeating this type of argument easily gives that  $\varphi$  is well defined. Moreover  $\varphi$  is easily seen to be a homomorphism.

Finally  $\text{Ker } \varphi = \{a \in P \mid s_1 = s_2 = \dots = 0\}$  i.e. if  $a \in \text{Ker } \varphi$  then

$$a_1 = r_1 d_1$$

$$a_2 = r_2 d_2 + r_1 x_2^1$$

$$a_3 = r_3 d_3 + r_2 x_3^2 + r_1 x_3^1, \quad \text{etc.}$$

i.e.  $a = \sum_{i=1}^{\infty} r_i x^i$  and so  $\text{Ker } \varphi = X$ .

Hence  $P/X \cong \prod \mathbb{Z}(d_i)$  where  $\mathbb{Z}(d_i)$  is to be interpreted as  $\mathbb{Z}$  if  $d_i = 0$ .

Given Proposition 2 one can easily recover the characterization of homomorphic images of  $P$  (Nunke [4] or Fuchs [2, Prop. 95.2]).

COROLLARY 3. Every epimorphic image of  $P$  is the direct sum of a cotorsion group and a direct product of infinite cyclic groups.

PROOF. Let  $K$  be a subgroup of  $P$  and let  $\bar{K}$  be the closure of  $K$  in the product topology. From Proposition 2,  $\bar{K}$  is a product, say  $\bar{K} = \sum_{i=1}^{\infty} * \langle x_i \rangle$  and  $P/\bar{K}$  is a product of cyclic groups. Let  $P = P_1 \oplus P_2$  where  $P_1, P_2$  are the products of the  $\langle e_n \rangle$  with  $d_n \neq 0$  and  $d_n = 0$  respectively. Then  $\bar{K} \leq P_1$  and  $P_1/\bar{K}$  is algebraically compact since it is a product of finite cyclic groups. Since  $\bigoplus_{i=1}^{\infty} \langle x_i \rangle$  is contained in  $K$ , the quotient  $\bar{K}/K$  is cotorsion and this combined with  $P_1/\bar{K}$  being cotorsion implies  $P_1/K$  is also cotorsion [1, 54 (D)].

#### REFERENCES

- [1] L. FUCHS, *Infinite Abelian Groups*, Vol. I, Academic Press, New York and London, 1970.
- [2] L. FUCHS, *Infinite Abelian Groups*, Vol. II, Academic Press, New York and London, 1973.
- [3] R. J. NUNKE, *On direct products of infinite cyclic groups*, Proc. Amer. Math. Soc., **13** (1962), pp. 66-71.
- [4] R. J. NUNKE, *Slender groups*, Acta Sci. Math. Szeged, **23** (1962), pp. 67-73.

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